# Mean first-passage time for an overdamped particle in a disordered force field

S. I. Denisov*
*Department of Mechanics and Mathematics, Sumy State University, 2, Rimskiy-Korsakov Street, 40007 Sumy, Ukraine*

Werner Horsthemke†
*Department of Chemistry, Southern Methodist University, Dallas, Texas 75275–0314*
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We derive a rigorous expression for the mean first-passage time of an overdamped particle subject to a constant bias in a force field with quenched disorder. Depending on the statistics of the disorder, the disorder-averaged mean first-passage time can undergo a transition from an infinite value for small bias to a finite value for large bias. This corresponds to a depinning transition of the particle. We obtain exact values for the depinning threshold for Gaussian disorder and also for a class of piecewise constant random forces, which we call generalized kangaroo disorder. For Gaussian disorder, we investigate how the correlations of the random force field affect the average motion of the particle. For kangaroo disorder, we apply the general results for the depinning transition to two specific examples, viz., dichotomous disorder and random fractal disorder.

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## I. INTRODUCTION

The motion of an overdamped particle in a potential models a broad variety of transport phenomena in physical, chemical, and biological systems. Ratchetlike mechanisms, the motion of a particle in an asymmetric periodic potential, have been proposed to explain the transport of large molecules in cells and through membranes [1]. The asymmetric potential can rectify symmetric nonequilibrium fluctuations, and give rise to fluctuation-induced transport [2], which may be used for the continuous sorting of macromolecules [3]. The transport properties are strongly perturbed by the presence of frozen disorder or defects in the ratchet potential [4]. The motion of an overdamped particle in a nonperiodic disorder potential is of interest in its own right. It provides a model for the dynamics of dislocations in solids and of domain walls in random-field magnets, for diffusion of test particles in porous media or turbulent flows, for electronic transport in amorphous media, and for other transport phenomena in random media [5,6]. It has also been used as a simple phenomenological model of glassy dynamics [7,8].

A distinctive feature of the dynamics in systems with random potentials, or quenched disorder, is the existence of the depinning transition. For certain statistics of the quenched disorder, the particle on average does not move below a threshold value of the external driving force, whereas it moves for a force above this value. In this paper, we study how thermal fluctuations interact with various types of quenched disorder, and derive expressions for the depinning threshold. The effects of randomness in driven systems are often nonintuitive, and it is therefore desirable to investigate model systems for which exact analytical results can be obtained. Previous studies investigated the motion of an overdamped particle in the presence of a Gaussian disorder potential [7,9]. We extend those studies by exploring specifically how the functional form of the disorder correlations effects the depinning transition. We then focus on a class of non-Gaussian random forces for which the depinning transition can be characterized analytically.

Earlier publications considered mainly spatially discrete situations, where the particle moves on a lattice [5,10–14]. The dynamics of the particle is described by a random walk with random hopping rates. We adopt a spatially continuous description in terms of a Langevin equation with quenched disorder forces. This point of view is better suited to situations where the disorder force field, or disorder potential, is well characterized, and hopping rates are a derivative quantity. Studies of random walks in random media [12–14] and of Langevin equations with quenched disorder [9,15–17] have shown that the mean first-passage time formalism is an effective tool to characterize the depinning transition. This formalism has the advantage that it does not rely on the periodic continuation of the random potential used in [7,11,18].

We consider a one-dimensional disorder potential. The equation of motion of an overdamped particle with coordinate $x$ in the presence of a random potential $U(x)$ and a constant external force $f$ is given by the Langevin equation

$$\dot{x}(t)=g(x(t))+f+\eta(t), \tag{1.1}$$

where $g(x)=-dU(x)/dx$. Without restriction of generality, we consider $f$ to be positive, i.e., the particle is driven to the right. The quenched disorder is described by the random force $g(x)$, which we assume to be a homogeneous, i.e., translationally invariant, random function with mean value zero,

$$\overline{g(x)}=0, \tag{1.2}$$

and correlation function $r(u)$,

$$\overline{g(y)g(x)}=r(|y-x|), \tag{1.3}$$

*Electronic address: denisov@ssu.sumy.ua
†Electronic address: whorsthe@mail.smu.edu

 

with $r(0)>0$ and $r(\infty)=0$. Here the overbar denotes averaging with respect to the random force field $g(x)$. The definition of $g(x)$ implies that

$$U(y)-U(x)=-\int_x^y g(s)ds \tag{1.4}$$

and

$$\begin{aligned}\overline{[U(y)-U(x)]^2}&=\int_x^y\int_x^y\overline{g(s)g(v)}ds\,dv\\&=\int_x^y\int_x^y r(s-v)ds\,dv\\&=2\int_0^{|y-x|}r(u)[|y-x|-u]du.\end{aligned} \tag{1.5}$$

Defining the correlator of the potential as usual by $K(z)\equiv\overline{[U(z+x)-U(x)]^2}=\overline{[U(z)-U(0)]^2}$, we obtain

$$K(z)=2\int_0^{|z|}r(u)[|z|-u]du. \tag{1.6}$$

Note that the potential $U(x)$, as an integral over a homogeneous random force, is itself not a homogeneous random function. The probability density of the random force $g(x)$ will be specified below. The particle is subject not only to spatial random disorder, but also to temporal random noise. We assume $\eta(t)$ to be Gaussian white noise,

$$\langle\eta(t)\rangle=0, \tag{1.7}$$

$$\langle\eta(t)\eta(t')\rangle=2\Delta\,\delta(t-t'), \tag{1.8}$$

where $\delta(t)$ is the $\delta$ function, and $\langle\;\rangle$ denotes averaging with respect to the noise $\eta(t)$. If the noise represents thermal equilibrium fluctuations, the white noise intensity $\Delta$ is equal to the absolute temperature of the heat bath.

We characterize the dynamical behavior of the particle by the disorder-averaged mean first-passage time $T=\overline{\langle t(0)\rangle}$, where $t(0)$ is the time a particle starting at $x(0)=0$ spends in the interval $(-\infty,L)$ before reaching the position $L>0$ for the first time, i.e., $t(0)$ is the so-called first-passage time. We choose the lower boundary to be at minus infinity to avoid finite-size effects, and to obtain the depinning threshold in the thermodynamic limit. The depinning transition corresponds to the transition from $T=\infty$ to $T<\infty$.

This paper is organized as follows. We derive the general expression for the mean first passage time $T$ in Sec. II. The role of the functional form of the correlations for the case of Gaussian disorder is analyzed in Sec. III. We consider generalizations of the kangaroo process, i.e., stepwise constant random forces, and derive analytical results for the depinning transition in Sec. IV. We conclude with a discussion of our results in Sec. V.

## II. MEAN FIRST-PASSAGE TIME

We denote a realization, or sample path, of the random force $g(x)$ by $\omega$: $g(\omega,x)$. The solution of the Langevin equation

$$\dot{x}(\omega,t)=g(\omega,x(t))+f+\eta(t) \tag{2.1}$$

is a Markovian diffusion process. Let $t(\omega,x)$ be the time that a particle starting at $x(\omega,0)=x$ with $a\leqslant x\leqslant b$ leaves the interval $(a,b)$ for the first time. For a given realization $\omega$ of the random force, the mean first-passage time $T(\omega,x)=\langle t(\omega,x)\rangle$ obeys the equation [19,20]

$$[g(\omega,x)+f]\frac{dT(\omega,x)}{dx}+\Delta\frac{d^2T(\omega,x)}{dx^2}=-1. \tag{2.2}$$

Since we are interested in the time when the particle reaches the upper boundary $b$ of the interval for the first time, no matter how often it has reached the lower boundary $a$, we consider $b$ to be an absorbing barrier and $a$ to be a reflecting barrier. The solution of Eq. (2.2) with these boundary conditions is given by [20]

$$T(\omega,x)=\frac{1}{\Delta}\int_x^b\frac{dy}{\psi(y)}\int_a^y\psi(z)dz,\quad a<b, \tag{2.3}$$

where

$$\psi(y)=\exp\left\{\frac{1}{\Delta}\int_a^y[f+g(\omega,\xi)]d\xi\right\}. \tag{2.4}$$

As mentioned in Sec. I, in our case $b=L$ and $a=-\infty$. Equation (2.3) remains valid as $a$ goes to $-\infty$, provided that $-\infty$ is a natural boundary [21] of the Markovian diffusion process $x(\omega,t)$ [19]. A diffusion process reaches a natural boundary with probability zero, even if time goes to infinity. A natural boundary has no effect on the mean first-passage time, and the result obtained for the depinning threshold is free of finite-size effects.

The boundary $a=-\infty$ is natural, if [21]

$$L_1(-\infty)=\int_{-\infty}^{\beta}\phi(x)dx=\infty, \tag{2.5}$$

where

$$\phi(x)=\exp\left\{-\frac{1}{\Delta}\int_\beta^x[f+g(\omega,z)]dz\right\}. \tag{2.6}$$

From Eq. (2.6) we obtain

$$\phi(x)=\exp\left\{\frac{f}{\Delta}(\beta-x)\left[1+\frac{1}{f(\beta-x)}\int_x^\beta g(\omega,z)dz\right]\right\}, \tag{2.7}$$

and therefore

$$\text{Prob}\{L_1(-\infty)=\infty\}=\text{Prob}\left\{\lim_{x\to-\infty}\left(1+\frac{1}{f(\beta-x)}\int_x^\beta g(\omega,z)dz\right)\geqslant 0\right\}. \tag{2.8}$$

Let us define the random variable $G(x;\beta)$ as

$$G(x;\beta)\equiv\frac{1}{\beta-x}\int_x^\beta g(\omega,z)dz. \tag{2.9}$$

The strong law of large numbers [22] implies that

$$\lim_{x\to-\infty} G(x;\beta)=\overline{g(z)}=0 \quad \text{(almost surely)}, \tag{2.10}$$

if and only if

$$\lim_{x\to-\infty} \overline{G(x;\beta)^2}=0. \tag{2.11}$$

Since

$$G(x;\beta)=\frac{1}{\beta-x}[U(x)-U(\beta)], \tag{2.12}$$

Eq. (2.11) is equivalent to

$$\lim_{x\to-\infty} \frac{1}{(\beta-x)^2}K(\beta-x)=0. \tag{2.13}$$

If the correlator of the potential is given by a power law, i.e., $K(z)\sim z^{\eta}$ for $z\to\infty$, then the boundary $a=-\infty$ is natural and Eq. (2.3) holds, if $\eta<2$. Defining the correlation length $\lambda$ of the random force $g(x)$ in the usual way,

$$\lambda\equiv\frac{1}{r(0)}\int_0^\infty |r(u)|du, \tag{2.14}$$

we find that a finite correlation length of the disorder force is a sufficient, although not necessary, condition for $a=-\infty$ to be natural.

From Eq. (2.3) we obtain

$$\begin{aligned} T(\omega,0)&=\frac{1}{\Delta}\int_0^L dy\,\exp\left\{-\frac{1}{\Delta}\int_{-\infty}^{y}[f+g(\omega,\xi)]d\xi\right\}\\ &\quad\times\int_{-\infty}^{y} dz\,\exp\left\{\frac{1}{\Delta}\int_{-\infty}^{z}[f+g(\omega,\xi)]d\xi\right\}\\ &=\frac{1}{\Delta}\int_0^L dy\int_{-\infty}^{y} dz\,\exp\left\{-\frac{f}{\Delta}(y-z)\right.\\ &\quad\left.-\frac{1}{\Delta}\int_z^y g(\omega,\xi)d\xi\right\}. \end{aligned} \tag{2.15}$$

Using the transformation of variables $x=(f/\Delta)(y-z)$, we find

$$T(\omega,0)=\frac{1}{f}\int_0^L dy\int_0^\infty dx\,\exp\left\{-x-\frac{1}{\Delta}\int_{y-(\Delta/f)x}^{y} g(\omega,\xi)d\xi\right\}, \tag{2.16}$$

and with $\zeta=(f/\Delta)(\xi-y)+x$,

$$\begin{aligned} T(\omega,0)&=\frac{1}{f}\int_0^L dy\int_0^\infty dx\,\exp\left\{-x-\frac{1}{f}\int_0^x g(\omega,(\Delta/f)\zeta\right.\\ &\quad\left.+y-(\Delta/f)x)d\zeta\right\}. \end{aligned} \tag{2.17}$$

Averaging over the disorder, we obtain the mean first-passage time

$$\begin{aligned} T&=\overline{T(\omega,0)}\\ &=\frac{1}{f}\int_0^L dy\int_0^\infty dx\,\exp(-x)\\ &\quad\times\overline{\exp\left\{-\frac{1}{f}\int_0^x g(\omega,(\Delta/f)\zeta+y-(\Delta/f)x)d\zeta\right\}}, \end{aligned} \tag{2.18}$$

Since $g(x)$ is a homogeneous random function, the mean first-passage time can be written as

$$T=T_0\int_0^\infty dx\,\exp(-x)\overline{\exp\left\{-\frac{1}{f}\int_0^x g(\Delta\zeta/f)d\zeta\right\}}, \tag{2.19}$$

where we have dropped the argument $\omega$, and $T_0=L/f$ is the mean first-passage time for $g(x)\equiv 0$. The Jensen inequality [23] [see also Eq. (4.18)], implies that

$$\begin{aligned} &\overline{\exp\left\{-\frac{1}{f}\int_0^x g(\Delta\zeta/f)d\zeta\right\}}\\ &\qquad\geqslant\exp\left\{-\frac{1}{f}\int_0^x \overline{g(\Delta\zeta/f)}d\zeta\right\}=1. \end{aligned} \tag{2.20}$$

Therefore $T\geqslant T_0$, i.e., quenched disorder always increases the mean first-passage time.

Finally, using the transformations of variables $x=fz/\Delta$ and $\xi=\Delta\zeta/f$ in Eq. (2.19), we obtain the general expression for the mean first-passage time:

$$T=\frac{L}{\Delta}\int_0^\infty dz\,\exp\left(-\frac{f}{\Delta}z\right)\overline{\exp\left\{-\frac{1}{\Delta}\int_0^z g(\xi)d\xi\right\}} \tag{2.21}$$

or

$$T=\frac{L}{\Delta}\int_0^\infty dz\,\exp\left(-\frac{f}{\Delta}z\right)\overline{\exp\left\{\frac{1}{\Delta}[U(z)-U(0)]\right\}}. \tag{2.22}$$

This result is equivalent to the expression for the inverse of the disorder-averaged velocity in Ref. [7] and agrees with the result for the mean first-passage time in Ref. [9].

To conclude this section, we derive the condition that the mean first-passage time is self-averaging in the large $L$ limit, i.e., that the disorder average is representative for a single realization of the quenched disorder. From Eq. (2.16) for the mean first-passage time for a given realization of the quenched disorder, we obtain

$$T(\omega,0)=\frac{L}{\Delta}\int_0^\infty dz\,\exp\left(-\frac{f}{\Delta}z\right)\Phi(\omega,z;L), \tag{2.23}$$

where

$$\Phi(\omega,z;L)=\frac{1}{L}\int_0^L dy\,\exp\left\{-\frac{1}{\Delta}\int_{y-z}^{y} g(\omega,\xi)d\xi\right\}. \tag{2.24}$$

Comparing Eqs. (2.23) and (2.21), we find that $T(\omega,0)$ is self-averaging, and that the intensive quantity $T(\omega,0)/L$ goes to the constant $T/L$ almost surely as $L$ goes to infinity, if $\Phi(\omega,z;L)$ is a self-averaging function, i.e., if

$$\lim_{L\to\infty}\Phi(\omega,z;L) = \overline{\exp\left\{-\frac{1}{\Delta}\int_{y-z}^{y} g(\omega,\xi)d\xi\right\}} \quad \text{(almost surely).} \tag{2.25}$$

[Recall that $g(x)$ is a homogeneous random function.] The strong law of large numbers [22] implies that Eq. (2.25) holds, if and only if

$$\lim_{L\to\infty}\overline{\Phi^2(\omega,z;L)} = \left(\overline{\exp\left\{-\frac{1}{\Delta}\int_{y-z}^{y} g(\omega,\xi)d\xi\right\}}\right)^2. \tag{2.26}$$

We have

$$\overline{\Phi^2(\omega,z;L)} = \frac{1}{L^2}\int_0^L dy\int_0^L dy'\,\overline{\exp[G(\omega,y,y',z)]}, \tag{2.27}$$

where

$$G(\omega,y,y',z) = -\frac{1}{\Delta}\int_{y-z}^{y} g(\omega,\xi)d\xi - \frac{1}{\Delta}\int_{y'-z}^{y'} g(\omega,\xi)d\xi. \tag{2.28}$$

If the random force field has a finite correlation length, then the dominant contribution to the integral on the right-hand side of Eq. (2.27) comes from regions where the two terms in $G(\omega,y,y',z)$ are uncorrelated. Therefore

$$\lim_{L\to\infty}\overline{\Phi^2(\omega,z;L)} = \lim_{L\to\infty}\frac{1}{L^2}\int_0^L dy\int_0^L dy'\,\overline{\exp\left\{-\frac{1}{\Delta}\int_{y-z}^{y} g(\omega,\xi)d\xi\right\}}\;\overline{\exp\left\{-\frac{1}{\Delta}\int_{y'-z}^{y'} g(\omega,\xi)d\xi\right\}}$$
$$= \left(\overline{\exp\left\{-\frac{1}{\Delta}\int_{y-z}^{y} g(\omega,\xi)d\xi\right\}}\right)^2, \tag{2.29}$$

and the mean first-passage time is self-averaging. A finite correlation length is a sufficient condition for $T(\omega,0)$ to be self-averaging, but not a necessary condition as we will show explicitly for Gaussian disorder at the end of Sec. III.

## III. GAUSSIAN DISORDER

As a first application of Eq. (2.21), we consider the case of Gaussian disorder. The evaluation of the average is facilitated by the following relation. If $G(x)$ is a Gaussian random function, then

$$\overline{\exp\{G(x)\}} = \exp\{(1/2)\overline{G(x)^2}\}. \tag{3.1}$$

The integral of a Gaussian random function is itself Gaussian, and

$$\overline{\exp\left\{-\frac{1}{\Delta}\int_0^z g(\xi)d\xi\right\}} = \exp\left\{\frac{1}{2\Delta^2}\int_0^z d\xi\int_0^z d\xi'\,\overline{g(\xi)g(\xi')}\right\}$$
$$= \exp\left\{\frac{1}{\Delta^2}\int_0^z dv\, r(v)(z-v)\right\}$$
$$= \exp\left\{\frac{1}{2\Delta^2}K(z)\right\}. \tag{3.2}$$

Thus, for Gaussian disorder, the mean first-passage time is given by

$$T = \frac{L}{\Delta}\int_0^\infty dz\,\exp\left\{-\frac{f}{\Delta}z + \frac{1}{\Delta^2}\int_0^z dv\, r(v)(z-v)\right\} \tag{3.3}$$

$$= \frac{L}{\Delta}\int_0^\infty dz\,\exp\left\{-\frac{f}{\Delta}z + \frac{1}{2\Delta^2}K(z)\right\}. \tag{3.4}$$

According to Eqs. (3.3) and (3.4), the mean first-passage time is finite, i.e., no pinning occurs, if

$$f\Delta > \lim_{z\to\infty}\frac{1}{z}\int_0^z dv\, r(v)(z-v) = \lim_{z\to\infty}\frac{1}{2z}K(z). \tag{3.5}$$

Since $K(z)\geqslant 0$, the following cases are possible.

(1) If $\lim_{z\to\infty}K(z)/2z=\infty$, then $T=\infty$ for any $f$. The particle is always pinned, no matter how large the driving force $f$.

(2) If $\lim_{z\to\infty}K(z)/2z=\chi\,(0<\chi<\infty)$, then $T=\infty$ for $f<\chi/\Delta$, and $T<\infty$ for $f>\chi/\Delta$. The particle experiences a depinning transition with the threshold value $f_c=\chi/\Delta$. If $f=\chi/\Delta$, then the mean first-passage time will be finite or infinite, depending on the form of the second term in the asymptotic expansion of $K(z)$.

(3) If $\lim_{z\to\infty}K(z)/2z=0$, then $T<\infty$ for any $f$. The particle will always move, even for a small but nonvanishing driving force $f$.

To understand the physical significance of these three cases, we determine the functional form of the correlation function $r(u)$ that gives rise to each case. Since $K(z)\geqslant 0$, the function $r(u)$ must satisfy the inequality

$$\int_0^z r(u)du \geq \frac{1}{z}\int_0^z ur(u)du \tag{3.6}$$

for any $z \geq 0$. This inequality implies that

$$R \equiv \int_0^\infty r(u)du \geq 0. \tag{3.7}$$

We begin by considering the case when the correlation function of the random force field $g(x)$ decays monotonically with distance. If $r(u) \geq 0$ for $u \geq 0$, then

$$\frac{1}{z}\int_0^z ur(u)du = \frac{a(z)}{z}\int_0^z r(u)du, \tag{3.8}$$

where $0 < a(z) < z$, and inequality (3.6) is always satisfied. In this case, $\lim_{z\to\infty} K(z)/2z = \infty$ if $R = \infty$, and $\lim_{z\to\infty} K(z)/2z = \chi$ if $R = \chi$. Indeed, according to Eqs. (1.6) and (3.8), we have

$$\frac{K(z)}{2z} = \left(1 - \frac{a(z)}{z}\right)\int_0^z r(u)du. \tag{3.9}$$

Since $0 < a(z) < z$, it follows that $\lim_{z\to\infty} K(z)/2z = \infty$ if $R = \infty$. If $0 < R < \infty$, then $\lim_{u\to\infty} ur(u) = 0$,

$$\lim_{z\to\infty} \frac{1}{z}\int_0^z ur(u)du = 0, \tag{3.10}$$

and $\lim_{z\to\infty} K(z)/2z = R$. Note that for $r(u) \geq 0$ the third case is not realized.

We now consider the case that the correlations in the random force field decay in an oscillatory manner. We limit our study to those situations where the correlation function $r(u)$ does not exhibit further sign changes beyond a certain distance $c > 0$. As for monotonically decreasing correlation functions, the first case is realized if $R = \infty$. Since

$$\begin{aligned}\frac{K(z)}{2z} &= \int_0^z r(u)du - \frac{1}{z}\int_0^z ur(u)du \\ &= \int_0^c r(u)du - \frac{1}{z}\int_0^c ur(u)du + \left(1 - \frac{b(z)}{z}\right)\int_c^z r(u)du,\end{aligned} \tag{3.11}$$

where $c < b(z) < z$, and since the condition $R = \infty$ implies

$$\int_c^\infty r(u)du = \infty, \tag{3.12}$$

we obtain $\lim_{z\to\infty} K(z)/2z = \infty$, neglecting the first two terms in Eq. (3.11) as $z \to \infty$. If $0 \leq R < \infty$, then $\lim_{u\to\infty} ur(u) = 0$, Eq. (3.10) holds, and therefore $\lim_{z\to\infty} K(z)/2z = R$.

These results show that for a Gaussian random force field whose correlation function either decreases monotonically or does not change sign beyond some $c > 0$, the particle is always pinned, if the correlation length of the quenched disorder is infinite. If the correlation length is finite, a depinning transition occurs at a finite threshold value for the driving force $f$, unless regions of positive and negative correlations in the force field are exactly balanced. In the latter case, the particle will move for any nonvanishing driving force.

The first case, where $R = \infty$, corresponds to a power-law behavior for the correlator of the potential, $K(z) \sim z^\eta$ as $z \to \infty$ with $1 < \eta < 2$, since

$$K(z) \sim z\int_c^z r(u)du, \tag{3.13}$$

and $r(\infty) = 0$. Thus the boundary at minus infinity is, as required, a natural boundary. For the second case, where $0 < R < \infty$, we find $K(z) = 2zR$ as $z \to \infty$, or $\eta = 1$. This case is realized for a broad class of random forces $g(x)$ that have a finite correlation length. In particular, the Sinai model, where the potential is a Wiener process and the random force field is $\delta$ correlated, belongs to this class. For the third case, where $R = 0$, we obtain $\eta < 1$. This case is realized for the class of random forces $g(x)$ that have a finite correlation length and satisfy the condition $R = 0$. A specific example is a random force field $g(x)$ with the correlation function

$$r(u) = r(0)\frac{1 - (\kappa - 2)u/\delta}{(1 + u/\delta)^\kappa}. \tag{3.14}$$

Here $\delta$ is a parameter that has the dimension of length, and $\kappa > 2$. The condition $\kappa > 2$ follows from the condition $\lambda < \infty$. For random forces with correlation functions of this form, we obtain

$$\begin{aligned}K(z) = 2\delta^2 r(0)\Bigg\{&\frac{1}{\kappa-3}\left[1 - \frac{1}{(1+z/\delta)^{\kappa-3}}\right] \\ &- \frac{1}{\kappa-2}\left[1 - \frac{1}{(1+z/\delta)^{\kappa-2}}\right]\Bigg\}\end{aligned} \tag{3.15}$$

if $\kappa \neq 3$, and

$$K(z) = 2\delta^2 r(0)\left[\ln(1 + z/\delta) + \frac{1}{1+z/\delta} - 1\right] \tag{3.16}$$

if $\kappa = 3$. Equation (3.15) shows that $\eta = 0$ for $\kappa > 3$, and $\eta = 3 - \kappa$ for $2 < \kappa < 3$. If $\kappa = 3$, then the correlator diverges logarithmically: $K(z) \sim \ln(z/\delta)$ as $z/\delta \to \infty$.

Note that for Gaussian disorder with a finite correlation length, the particle is always pinned in the absence of thermal noise. The certain pinning in a noiseless system is caused by realizations with $g(\omega, x'(\omega)) = -f$ with $x'(\omega) \in (0, L)$, which occur with nonzero probability. For such realizations, the disorder force overcomes the external bias force, and the particle will be trapped, if the noise intensity $\Delta$ vanishes. Therefore the mean first-passage time is always infinite for Gaussian disorder in the absence of noise; no depinning transition occurs without the help of the temporal noise. For finite noise intensity and Gaussian disorder with a finite correlation length, the depinning transition occurs at $f = f_c$, where

$$f_c = R/\Delta. \tag{3.17}$$

If $f > f_c$, the mean first-passage time is a monotone decreas-

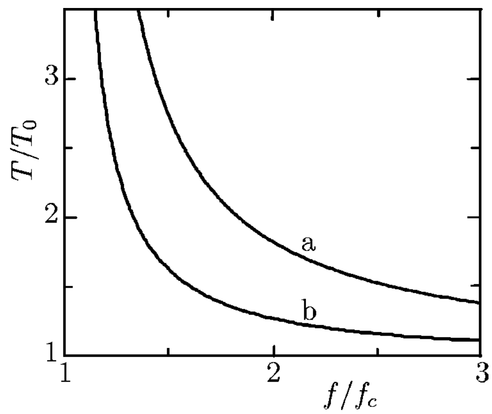


FIG. 1. Dependence of $T/T_0$ on $f/f_c$ for Gaussian disorder with $r(u)=r(0)\exp(-u/\lambda)$ and $\varphi=0.1$ (a), $\varphi=1$ (b).

ing function of $f$ that tends to zero as $f\to\infty$. To illustrate the behavior of $T$ as a function of $f$, we consider Gaussian disorder with $r(u)=r(0)\exp(-u/\lambda)$. In this case, $K(z)=2r(0)\lambda[z-\lambda+\lambda\exp(-z/\lambda)]$, and $f_c=r(0)\lambda/\Delta$. The depinning threshold value of $f$ is directly proportional to the correlation length $\lambda$, and Eq. (3.4) yields

$$T=T_0\int_0^\infty dx\,\exp\left\{-x+\frac{f_c}{f}x-\varphi+\varphi\exp\left[-\frac{f_c}{f\varphi}x\right]\right\}, \tag{3.18}$$

where $\varphi=f_c\lambda/\Delta$. The dependence of $T/T_0$ on $f/f_c$ is shown in Fig. 1 for two values of the parameter $\varphi$.

To conclude this section, we verify explicitly that condition (2.26) holds for Gaussian disorder. In this case the right-hand side of Eq. (2.26) is given by

$$\left(\overline{\exp\left\{-\frac{1}{\Delta}\int_{y-z}^{y}g(\xi)d\xi\right\}}\right)^2=\exp\left[\frac{1}{\Delta^2}K(z)\right]. \tag{3.19}$$

Since $\overline{\exp[G(\omega,y,y',z)]}=\exp[\overline{G^2(\omega,y,y',z)}/2]$, and

$$\overline{G^2(\omega,y,y',z)}=\frac{2}{\Delta^2}\int_0^z dx\int_0^z dx'\,r(y-y'+x-x')+\frac{2}{\Delta^2}K(z), \tag{3.20}$$

for the left-hand side of Eq. (2.26) we obtain

$$\overline{\Phi^2(\omega,z;L)}=\frac{2}{L^2}\int_0^L dv\,\exp\left\{\frac{1}{\Delta^2}[K(z)+\Gamma(v;z)]\right\}(L-v), \tag{3.21}$$

where

$$\Gamma(v;z)=\int_0^z dx\int_0^z dx'\,r(v+x-x'). \tag{3.22}$$

Therefore, condition (2.26) is equivalent to

$$\lim_{L\to\infty}\frac{2}{L^2}\int_0^L dv\,\exp[\Gamma(v;z)/\Delta^2](L-v)=1. \tag{3.23}$$

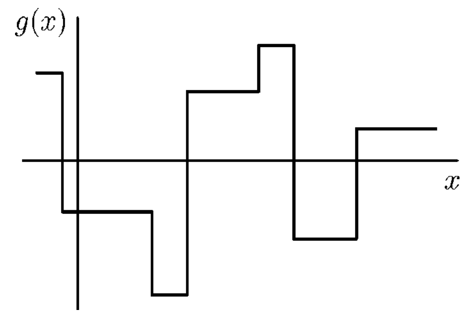


FIG. 2. Part of a sample path for generalized kangaroo disorder with $w_{-1}(g)=w_{+1}(g)$.

Condition (3.23) holds if and only if $r(v)\to 0$ as $v\to\infty$. (Note that this a weaker condition than requiring a finite correlation length for the disorder.) If the correlation function of the Gaussian disorder decays to zero as distance goes to infinity, then the mean first-passage time is self-averaging, and the disorder average is representative for a single realization.

## IV. KANGAROO DISORDER

To gain further insight into how the interplay of temporal noise and quenched disorder causes the depinning transition, we consider a class of stepwise constant random functions that are generalizations of the kangaroo process [24]. These random functions are homogeneous, and they are defined by a pair of probability densities: the jump density $p_\gamma(s)$ that a step has a length $s$, and the probability density $w_\gamma(g)$ that the step has a constant value $g$. To allow for the possibility that adjacent steps have different statistics, we index both probability densities by $\gamma$, with $\gamma=\pm1$. Realizations of this generalized kangaroo process (see Fig. 2) are constructed as follows. If the realization has a jump at $x=x_j$ and is in state $\gamma'$ before the jump, then the next jump occurs at $x=x_j+s$, where the probability density of $s$ is $p_\gamma(s)$ with $\gamma=-\gamma'$. The value of $g(x)$ is constant, $g$, on the interval $(x_j,x_j+s]$, and the probability density of $g$ is given by $w_\gamma(g)$. If $w_{-1}(g)=w_{+1}(g)$ and if $p_{-1}(s)=p_{+1}(s)=(1/\lambda)\exp(-s/\lambda)$, then $g(x)$ is the ordinary kangaroo process.

To determine the mean first-passage time for kangaroo disorder, we must evaluate the disorder average of $\exp\{-(1/\Delta)\int_0^z g(x)dx\}$. First consider realizations of the disorder that (i) have no jump in the interval $(0,z)$, i.e., the right edge $x_r$ of the step is larger than $z$; that (ii) are of type $\gamma$; and that (iii) have a value $g$. Those realizations occur with probability

$$dW_0^\gamma=P_\gamma w_\gamma(g)dg\int_z^\infty ds\,\tilde{p}_\gamma(s). \tag{4.1}$$

Here

$$P_\gamma=\int_0^\infty ds\,s\,p_\gamma(s)\left[\int_0^\infty ds\,s\sum_\gamma p_\gamma(s)\right]^{-1} \tag{4.2}$$

is the probability that a point $x$ belongs to a step of type $\gamma$, and

$$\tilde{p}_\gamma(s)=\int_s^\infty dr\frac{1}{r}p_\gamma(r) \tag{4.3}$$

is the probability density that $x_r=s$.

The probability $dW_n^\gamma$ that on the interval $(0,z)$ (i) the random force $g(x)$ has $n$ jumps at points $s_i$ belonging to the infinitesimal intervals $ds_i$ $(i=1,\ldots,n)$, (ii) the first step is of type $\gamma$, and (iii) $g(x)=\mathrm{const}\in(g_i,g_i+dg_i)$, is given by

$$dW_n^\gamma=P_\gamma\left(\prod_{i=1}^{n+1}dg_i w_{(-1)^{i-1}\gamma}(g_i)\right)\left(\prod_{i=2}^{n}ds_i p_{(-1)^{i-1}\gamma}(s_i)\right)\times ds_1\tilde{p}_\gamma(s_1)\int_{s_{n+1}}^\infty dsp_{(-1)^n\gamma}(s), \tag{4.4}$$

where $s_{n+1}=z-s_1-s_2-\cdots-s_n$. Since the probabilities $dW_0^\gamma$ and $dW_n^\gamma$ satisfy the normalization condition

$$\sum_\gamma\int_{-\infty}^{\infty}dW_0^\gamma+\sum_\gamma\sum_{n=1}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\underbrace{\int_{s_1\ge0}\cdots\int_{s_n\ge0}}_{s_1+\ldots+s_n\le z}dW_n^\gamma=1, \tag{4.5}$$

and since

$$\int_0^z g(\xi)d\xi=\sum_{i=1}^{n+1}g_is_i, \tag{4.6}$$

we obtain

$$\overline{\exp\left\{-\frac{1}{\Delta}\int_0^z g(\xi)d\xi\right\}}=\sum_\gamma P_\gamma\int_z^\infty ds\int_{-\infty}^{\infty}dgw_\gamma(g)\tilde{p}_\gamma(s)\exp\left[-\frac{gz}{\Delta}\right]+\sum_\gamma\sum_{n=1}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\underbrace{\int_{s_1\ge0}\cdots\int_{s_n\ge0}}_{s_1+\ldots+s_n\le z}dW_n^\gamma\exp\left[-\frac{1}{\Delta}\sum_{i=1}^{n+1}g_is_i\right]. \tag{4.7}$$

Using Eq. (4.7), we can write Eq. (2.21) in the form

$$T=\frac{L}{\Delta}\sum_\gamma\Bigg\{P_\gamma\int_0^\infty dz\int_z^\infty ds\int_{-\infty}^\infty dgw_\gamma(g)\tilde{p}_\gamma(s)\exp\left[-\frac{g+f}{\Delta}z\right]+\sum_{n=1}^{\infty}\int_0^\infty dz\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\underbrace{\int_{s_1\ge0}\cdots\int_{s_n\ge0}}_{s_1+\ldots+s_n\le z}dW_n^\gamma\exp\left[-\frac{1}{\Delta}\sum_{i=1}^{n+1}g_is_i-\frac{fz}{\Delta}\right]\Bigg\}. \tag{4.8}$$

Let us call the second term of the right hand side of Eq. (4.8), $Y_\gamma$. Using the $\delta$ function $\delta(z-\Sigma_{i=1}^{n+1}s_i)$, we can rewrite this as

$$Y_\gamma=\sum_{n=1}^{\infty}\int_0^\infty dz\int_{-\infty}^\infty\cdots\int_{-\infty}^\infty\int_0^\infty\cdots\int_0^\infty ds_{n+1}dW_n^\gamma\delta\left(z-\sum_{i=1}^{n+1}s_i\right)\exp\left[-\frac{1}{\Delta}\sum_{i=1}^{n+1}g_is_i-\frac{fz}{\Delta}\right]$$
$$=\sum_{n=1}^{\infty}\int_{-\infty}^\infty\cdots\int_{-\infty}^\infty\int_0^\infty\cdots\int_0^\infty ds_{n+1}dW_n^\gamma\exp\left[-\frac{1}{\Delta}\sum_{i=1}^{n+1}(g_i+f)s_i\right]. \tag{4.9}$$

Substituting Eq. (4.4) into Eq. (4.9) and using the simple identity

$$\sum_{n=1}^{\infty}a_n=\sum_{k=1}^{\infty}a_{2k-1}+\sum_{k=1}^{\infty}a_{2k}, \tag{4.10}$$

we find

$$Y_\gamma=P_\gamma G_{-\gamma}\tilde{R}_\gamma\sum_{k=1}^{\infty}R_\gamma^{k-1}R_{-\gamma}^{k-1}+P_\gamma G_\gamma\tilde{R}_\gamma\sum_{k=1}^{\infty}R_\gamma^{k-1}R_{-\gamma}^{k}, \tag{4.11}$$

where

$$G_\gamma \equiv \int_0^\infty dz \int_z^\infty ds \int_{-\infty}^\infty dg\, w_\gamma(g) p_\gamma(s) \exp\left[-\frac{g+f}{\Delta} z\right], \tag{4.12}$$

$$R_\gamma \equiv \int_0^\infty ds \int_{-\infty}^\infty dg\, w_\gamma(g) p_\gamma(s) \exp\left[-\frac{g+f}{\Delta} s\right], \tag{4.13}$$

$$\tilde{R}_\gamma \equiv \int_0^\infty ds \int_{-\infty}^\infty dg\, w_\gamma(g) \tilde{p}_\gamma(s) \exp\left[-\frac{g+f}{\Delta} s\right]. \tag{4.14}$$

Using Eq. (4.11) in the expression for the disorder-averaged mean first-passage time [Eq. (4.8)], and defining

$$\tilde{G}_\gamma \equiv \int_0^\infty dz \int_z^\infty ds \int_{-\infty}^\infty dg\, w_\gamma(g) \tilde{p}_\gamma(s) \exp\left[-\frac{g+f}{\Delta} z\right], \tag{4.15}$$

for generalized kangaroo disorder we obtain

$$T = \frac{L}{\Delta} \sum_\gamma P_\gamma \left[\tilde{G}_\gamma + \tilde{R}_\gamma (G_{-\gamma} + G_\gamma R_{-\gamma}) \sum_{k=1}^\infty R_\gamma^{k-1} R_{-\gamma}^{k-1}\right]. \tag{4.16}$$

The mean first-passage time for generalized kangaroo disorder changes from $T=\infty$ to $T<\infty$, i.e., the depinning transition occurs, as $R_\gamma R_{-\gamma} = R_{+1}R_{-1}$ changes from $R_{+1}R_{-1} > 1$ to $R_{+1}R_{-1} < 1$. For the latter case, i.e., when the particle is not pinned, Eq. (4.16) simplifies to

$$T = \frac{L}{\Delta} \sum_\gamma P_\gamma \left[\tilde{G}_\gamma + \tilde{R}_\gamma \frac{G_{-\gamma} + G_\gamma R_{-\gamma}}{1 - R_\gamma R_{-\gamma}}\right]. \tag{4.17}$$

We find a lower bound for $R_{+1}R_{-1}$ using the Jensen inequality [23]

$$M_h\{f(h)\} \geqslant f(M_h\{h\}). \tag{4.18}$$

Here $h$ is a random variable with probability density $p(h)$, $f(h)$ is a convex function of $h$, and $M_h\{f(h)\}$ is the average, or mathematical expectation, i.e,

$$M_h\{f(h)\} = \int dh\, f(h) p(h). \tag{4.19}$$

With $\lambda_\gamma = M_s^\gamma\{s\}$ and $g_\gamma = M_g^\gamma\{g\}$, from Eqs. (4.13) and (4.18) we obtain

$$\begin{aligned} R_\gamma &= M_g^\gamma \left\{ M_s^\gamma \left\{ \exp\left[-\frac{g+f}{\Delta} s\right]\right\}\right\} \\ &\geqslant M_g^\gamma \left\{ \exp\left[-\frac{g+f}{\Delta} \lambda_\gamma\right]\right\} \\ &\geqslant \exp\left[-\frac{g_\gamma + f}{\Delta} \lambda_\gamma\right]. \end{aligned} \tag{4.20}$$

The condition $\overline{g(x)} = 0$ implies that $g_{+1}\lambda_{+1} + g_{-1}\lambda_{-1} = 0$, and Eq. (4.20) yields

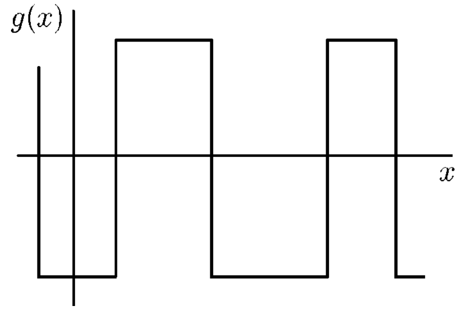


FIG. 3. Part of a sample path for dichotomous disorder.

$$R_{+1}R_{-1} \geqslant \exp\left[-\frac{\lambda_{+1} + \lambda_{-1}}{\Delta} f\right]. \tag{4.21}$$

The equality in Eq. (4.18) holds if and only if $f(h)$ is a linear function or $h=$const. This implies that for $\lambda_{+1} + \lambda_{-1} > 0$, Eq. (4.21) is reduced to the strict inequality

$$R_{+1}R_{-1} > \exp\left[-\frac{\lambda_{+1} + \lambda_{-1}}{\Delta} f\right]. \tag{4.22}$$

(The case $\lambda_{+1} + \lambda_{-1} = 0$ is singular and is considered in Sec. IV B.) Strict inequality (4.22) implies that if $R_{+1}R_{-1} < 1$ for a certain value of $f$, then there exists a value $f_c$ such that as $f$ is decreased below this value, the condition $R_{+1}R_{-1} < 1$ ceases to hold. In order words, for kangaroo disorder the particle always experiences a pinning transition as $f$ decreases. The third case, which occurs for Gaussian disorder, is not realized. We now consider some instructive examples of kangaroo disorder.

### A. Dichotomous disorder

The generalized kangaroo process reduces to a random telegraph signal or Markovian dichotomous noise [21], if $w_\gamma(g) = \delta(g - \gamma g_0)$ and $p_\gamma(s) = (1/\lambda)\exp(-s/\lambda)$. A sample path is shown in Fig. 3. Straightforward calculations yield

$$R_\gamma = \frac{\Delta}{\Delta + \lambda(f + \gamma g_0)}, \tag{4.23}$$

$$G_\gamma = \frac{\Delta\lambda}{\Delta + \lambda(f + \gamma g_0)}, \tag{4.24}$$

$$\tilde{R}_\gamma = \frac{\Delta}{\lambda(f + \gamma g_0)} \ln\left[1 + \frac{\lambda(f + \gamma g_0)}{\Delta}\right], \tag{4.25}$$

$$\tilde{G}_\gamma = \frac{\Delta}{f + \gamma g_0} - \frac{\Delta^2}{\lambda(f + \gamma g_0)^2} \ln\left[1 + \frac{\lambda(f + \gamma g_0)}{\Delta}\right] \tag{4.26}$$

if $\Delta + \lambda(f - g_0) > 0$. We rewrite these expressions in a more compact form by introducing $\sigma = \lambda f/\Delta$ and $\rho = \lambda g_0/\Delta$:

$$R_{\pm 1} = \frac{1}{1 + \sigma \pm \rho}, \quad \tilde{R}_{\pm 1} = \frac{1}{\sigma \pm \rho} \ln(1 + \sigma \pm \rho), \tag{4.27}$$

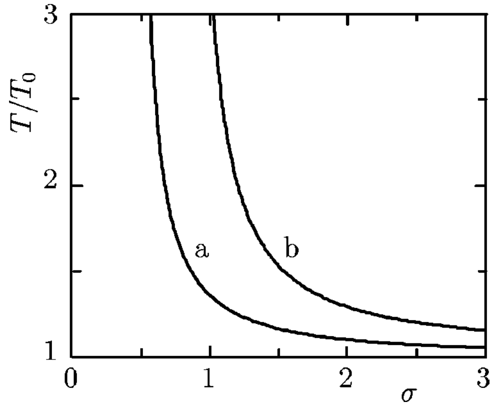


FIG. 4. Dependence of $T/T_0$ on $\sigma$ for dichotomous disorder with $\rho=1$ (a) and $\rho=1.5$ (b). In the first case, the particle moves with assistance of the temporal noise if $0.414<\sigma<1$, and in the second case does so if $0.803<\sigma<1.5$.

$$G_{\pm 1}=\lambda R_{\pm 1}, \quad \tilde{G}_{\pm 1}=\frac{\lambda}{\sigma\pm\rho}-\frac{\lambda}{(\sigma\pm\rho)^2}\ln(1+\sigma\pm\rho). \tag{4.28}$$

If $R_{+1}R_{-1}<1$, the mean first-passage time is finite, and is given by

$$T=T_0\frac{\sigma\rho}{\sigma^2+2\sigma-\rho^2}\left[\frac{1+\sigma+\rho}{(\sigma+\rho)^2}\ln(1+\sigma+\rho)-\frac{1+\sigma-\rho}{(\sigma-\rho)^2}\ln(1+\sigma-\rho)\right]+T_0\frac{\sigma^2}{\sigma^2-\rho^2}, \tag{4.29}$$

since $P_{+1}=P_{-1}=1/2$. The condition $R_{+1}R_{-1}<1$ is equivalent to $\sigma^2+2\sigma-\rho^2>0$, or $\sigma>-1+\sqrt{1+\rho^2}$. This implies that for dichotomous quenched disorder the particle moves for forces larger than the threshold value

$$f_c=-\Delta/\lambda+\sqrt{(\Delta/\lambda)^2+g_0^2}. \tag{4.30}$$

The depinning threshold value $f_c$ is a nonlinear function of $\lambda$, which goes to $f_c=(g_0^2/2\Delta)\lambda$ as $\lambda\to 0$, and to $f_c=g_0-\Delta/\lambda$ as $\lambda\to\infty$. As expected, the particle moves for any bias force $f$ that exceeds the magnitude of the disorder forces $g_0$. However, even if $f$ is less than $g_0$, the particle can avoid being pinned and move with assistance from the temporal noise. This statement is illustrated in Fig. 4, where the dependence of $T/T_0$ on $\sigma$ is shown for $\rho=1$ and 1.5. For $f<g_0$, the depinning threshold in terms of the white noise intensity is given by

$$\Delta_c=\frac{\lambda}{2f}(g_0^2-f^2). \tag{4.31}$$

This expression shows that the depinning threshold value of the noise intensity is directly proportional to the correlation length of the quenched disorder.

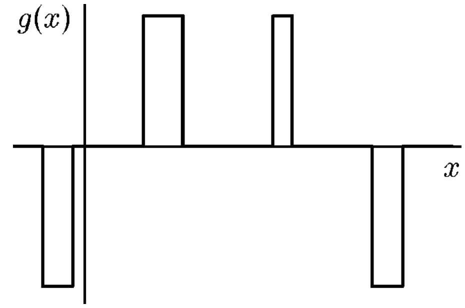


FIG. 5. Part of a sample path for prefractal disorder with $\lambda_{+1}>\lambda_{-1}$.

### B. Fractal disorder

Disorder often has a self-similar or fractal structure [25,26]. Here we study a simple example of fractal disorder arising from the kangaroo process defined by

$$w_{+1}(g)=\delta(g), \quad w_{-1}(g)=\frac{1}{2}[\delta(g-g_0)+\delta(g+g_0)], \tag{4.32}$$

$$p_{\pm 1}(s)=\frac{1}{\lambda_{\pm 1}}\exp\left(-\frac{s}{\lambda_{\pm 1}}\right). \tag{4.33}$$

A realization of such a process, which we call prefractal disorder, is shown in Fig. 5. For $\Delta+\lambda_{-1}(f-g_0)>0$, Eqs. (4.12)–(4.15) yield

$$\begin{aligned} R_{+1}&=\frac{1}{1+\sigma_+}, \quad R_{-1}=\frac{1+\sigma_-}{(1+\sigma_-)^2-\rho_-^2}, \\ \tilde{R}_{+1}&=\frac{1}{\sigma_+}\ln(1+\sigma_+), \\ \tilde{R}_{-1}&=\frac{1}{2(\sigma_-+\rho_-)}\ln(1+\sigma_-+\rho_-)+\frac{1}{2(\sigma_--\rho_-)}\ln(1+\sigma_--\rho_-), \\ G_{+1}&=\lambda_{+1}R_{+1}, \quad G_{-1}=\lambda_{-1}R_{-1}, \\ \tilde{G}_{+1}&=\frac{\lambda_{+1}}{\sigma_+}-\frac{\lambda_{+1}}{\sigma_+^2}\ln(1+\sigma_+), \\ \tilde{G}_{-1}&=\frac{\lambda_{-1}\sigma_-}{\sigma_-^2-\rho_-^2}-\frac{\lambda_{-1}}{2(\sigma_-+\rho_-)^2}\ln(1+\sigma_-+\rho_-)-\frac{\lambda_{-1}}{2(\sigma_--\rho_-)^2}\ln(1+\sigma_--\rho_-). \end{aligned} \tag{4.34}$$

Here $\sigma_{\pm}=\lambda_{\pm 1}f/\Delta$ and $\rho_-=\lambda_{-1}g_0/\Delta$. Taking into account $P_{+1}=1-P_{-1}=\lambda_{+1}/(\lambda_{+1}+\lambda_{-1})$, we obtain for the mean first-passage time [Eq. (4.17)] in the case of prefractal disorder

$$T=T_0\frac{(\sigma_++\sigma_-)\sigma_-^2-\sigma_+\rho_-^2}{(\sigma_++\sigma_-)(\sigma_-^2-\rho_-^2)}+T_0\frac{\rho_-^2(1+\sigma_+)\ln(1+\sigma_+)}{(\sigma_++\sigma_-)[(1+\sigma_-)(\sigma_++\sigma_-+\sigma_+\sigma_-)-(1+\sigma_+)\rho_-^2]}$$
$$+\frac{T_0}{2}\frac{\sigma_-\rho_-[\sigma_+(1+\sigma_--\rho_-)+\sigma_-](1+\sigma_-+\rho_-)\ln(1+\sigma_-+\rho_-)}{(\sigma_++\sigma_-)(\sigma_-+\rho_-)^2[(1+\sigma_-)(\sigma_++\sigma_-+\sigma_+\sigma_-)-(1+\sigma_+)\rho_-^2]}$$
$$-\frac{T_0}{2}\frac{\sigma_-\rho_-[\sigma_+(1+\sigma_-+\rho_-)+\sigma_-](1+\sigma_--\rho_-)\ln(1+\sigma_--\rho_-)}{(\sigma_++\sigma_-)(\sigma_--\rho_-)^2[(1+\sigma_-)(\sigma_++\sigma_-+\sigma_+\sigma_-)-(1+\sigma_+)\rho_-^2]}. \quad (4.35)$$

Equation (4.35) holds if $R_{+1}R_{-1}<1$, i.e., if

$$(1+\sigma_-)(\sigma_++\sigma_-+\sigma_+\sigma_-)>(1+\sigma_+)\rho_-^2. \quad (4.36)$$

We now consider the case of fractal disorder where $\lambda_{+1}=l(\lambda_{-1}/l)^\alpha$, $g_0=\tilde{g}(l/\lambda_{-1})^\epsilon$, with $\lambda_{-1}\to 0$ and $0<\alpha<1$. (Here $l$ and $\tilde{g}$ are parameters which have dimensions of $\lambda_{\pm 1}$ and $g_0$, respectively.) To obtain a nontrivial limit for the disorder force, we impose $\epsilon>0$. The random set on the $x$ axis defined by the condition $g(x)\neq 0$ is a fractal set, and its Hausdorff or fractal dimension is $d_H=\alpha$. This follows from the definition of $d_H$,

$$d_H=\lim_{\lambda_{-1}\to 0}\frac{\ln(\overline{S}/\lambda_{-1})}{\ln(1/\lambda_{-1})}, \quad (4.37)$$

where $\overline{S}$ is the average total length of the intervals for which $g(x)\neq 0$ in some interval $S$ on the $x$ axis. The average lengths of the intervals with $g(x)=0$ and with $g(x)\neq 0$ are $\lambda_{+1}$ and $\lambda_{-1}$, respectively. Therefore, $\overline{S}=S\lambda_{-1}/(\lambda_{+1}+\lambda_{-1})$, and for $\lambda_{-1}\to 0$ we have $\overline{S}\sim\lambda_{-1}^{1-\alpha}$; Eq. (4.37) then yields $d_H=\alpha$.

For $\lambda_{-1}\to 0$, condition (4.36) holds if (i) $2-2\epsilon-\alpha>0$, or (ii) $2-2\epsilon-\alpha=0$ and $f>f_c$. Here $f_c=\tilde{g}^2 l/\Delta$ is the depinning threshold. In the first case, Eq. (4.35) simplifies to $T=T_0$, and in the second case to $T=L/(f-f_c)$ for $f>f_c$. If $2-2\epsilon-\alpha<0$, then $T=\infty$ for all values of $f$. We conclude that if $\epsilon<1-\alpha/2$, the particle moves for any nonvanishing force $f$. In this case, the fractal disorder forces do not tend to infinity fast enough on the fractal to pin the particle. A depinning transition occurs if $\epsilon=1-\alpha/2$; the particle is pinned for $f<\tilde{g}^2 l/\Delta$. If the fractal disorder forces go to infinity too fast, i.e., $\epsilon>1-\alpha/2$, the particle is always localized by the fractal disorder.

## V. CONCLUSIONS

We have used the mean first-passage time formalism to obtain exact analytical results for the depinning transition of an overdamped particle in a random force field. This formalism is applicable if the lower boundary of the state space is a natural boundary. We have shown that this is the case if the correlation function of the random force decreases with distance. A finite correlation length of the quenched disorder is a sufficient, but not necessary, condition for $-\infty$ to be natural. We have also shown that a finite correlation length is a sufficient, but not necessary, condition for the mean first-passage time to be self-averaging in the limit of infinite system size. For Gaussian disorder, it turns out that a finite correlation length is a necessary condition for the particle to undergo a depinning transition. If the correlation length is infinite, then the particle is localized on average, no matter how strong the constant bias force.

While Gaussian disorder was studied before [7,9], exact results for other types of random forces were not known. The main contribution of this work is the derivation of exact expressions for the depinning threshold in the case of generalized kangaroo disorder. A broad variety of disorders can approximately be described by such disorder. For example, it allows us to model random fractal disorder. We have shown that for kangaroo disorder, the particle will always experience a pinning transition as the constant bias force decreases. Piecewise constant random forces on average will always localize the particle for sufficiently weak external forcing, in contrast to Gaussian disorder.

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